Structure of wave functions in (1+2)-body random matrix ensembles

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Random matrix ensembles defined by a mean-field one body plus a chaos generating random two-body interaction [called embedded ensembles of $(1+2)$-body interactions] predict for wave functions, in the chaotic domain, an essentially one-parameter Gaussian forms for the energy dependence of the number of principal components (NPC) and the localization length l_H (defined by information entropy), which are two important measures of chaos in finite interacting many-particle systems. Numerical embedded ensemble calculations and nuclear shell-model results, for NPC and l_H, are compared with the theory. These analyses clearly point out that for realistic finite interacting many-particle systems, in the chaotic domain, wave-function structure is given by $(1+2)$-body embedded random matrix ensembles.

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I. INTRODUCTION

In the last few years, the study of quantum chaos in isolated finite interacting particle systems has shifted from spectral statistics to properties of wave functions and transition strengths (for example, electromagnetic and Gamow-Teller transition strengths in atomic nuclei, dipole strengths in atoms, etc.). Working in this direction, several research groups have recognized recently that the two-body random matrix ensembles and their various extended versions form good models for understanding various aspects of chaos in interacting particle systems [1]. In particular, using the so-called embedded Gaussian orthogonal ensemble of $(1+2)$-body interactions [EGOE(1+2)] defined by a mean-field one body plus a chaos generating random two-body interaction, there are now several studies on the nature of occupancies of single-particle states, strength functions (or local density of states), information entropy, transition strength sums, and transition-matrix elements of one-body transition operators, Fock-space localization, etc., in the chaotic domain of interacting particle systems such as atoms [2], nuclei [1,3], quantum dots [4], quantum computers [5], and so on. Reference [1] gives a recent overview of this subject. The focus in the present article is on two important measures of localization (in wave functions and transition strength distributions): (i) number of principal components (NPC) [or the inverse participation ratio (IPR)]; (ii) localization length l_H as defined by the information entropy (S^{info}). It is well established that the NPC in wave functions characterizes various layers of chaos in interacting particle systems [6]. In addition, for systems such as atomic nuclei, NPC for transition strengths is a measure of fluctuations in transition strength sums. Similarly, the role of l_H in quantum chaos studies is well emphasized by Izrailev [7] and more significantly, using nuclear physics examples [8], it is well demonstrated that the wave-function entropy S^{info} coincides with the thermodynamic entropy for many-particle systems with two-body interactions of sufficient strength but only in the presence of a mean field, i.e., in the chaotic domain but with a mean field — therefore, the significance of EGOE(1+2). Clearly, deriving the predictions of EGOE(1+2) for NPC and l_H are of considerable importance. This problem was addressed in Refs. [9,10]. In [9] results for NPC in wave functions, in the so-called Breit-Wigner (BW) domain, are derived. On the other hand, in [10] results in the so-called Gaussian domain (the BW and Gaussian domains are defined in Sec. II ahead) are derived for NPC and l_H in transition strength distributions with only the final results mentioned for wave functions. The purpose of the present paper is to give a detailed derivation of the results mentioned in [10] for NPC and l_H in wave functions and subject them to numerical tests. Now we will give a preview.

Section II gives some of the basic results for EGOE (1+2). In Sec. III, formulas for NPC and l_H in wave functions are derived by exploiting the Gaussian nature and the associated properties of strength functions in EGOE(1+2). Numerical tests of the theory are given in Sec. IV. Finally, Sec. V gives concluding remarks.

II. BASIC RESULTS FOR (1+2)-BODY RANDOM MATRIX ENSEMBLES

Given m fermions in N single-particle states, assuming at the outset that the many-particle spaces are direct product spaces of the single-particle states, two-body random matrix ensembles (usually called TBRE) are generated by defining the Hamiltonian H, which is two-body, to be a random matrix in the two-particle spaces and then propagating it to the $\binom{N}{m}$ dimensional m-particle spaces by using their geometry (direct product structure); often one considers a GOE representation in the two-particle spaces and then the TBRE is called EGOE(2); see [1] for more details. For a EGOE(2), with $N \gg m \gg 2$, the normalized state density $\rho(E)=\langle \delta(H-E)\rangle$ takes Gaussian form and it is defined by its centroid $\epsilon=\langle H\rangle$ and variance $\sigma^2=\langle (H-\epsilon)^2\rangle$. In order to explicitly state that the state density is generated by the Hamiltonian H, sometimes $\rho(E)$ is denoted as $\rho^H(E)$ and similarly ϵ as ϵ_H and σ as σ_H. Note that the averages $\langle r\rangle$ are over the m-particle spaces; in the nuclear physics examples, they are usually over the m-particle spaces with fixed angular momentum (J) and isospin (T) which are good quantum numbers. Just as the state density, given a transition operator $\mathcal{O}$,

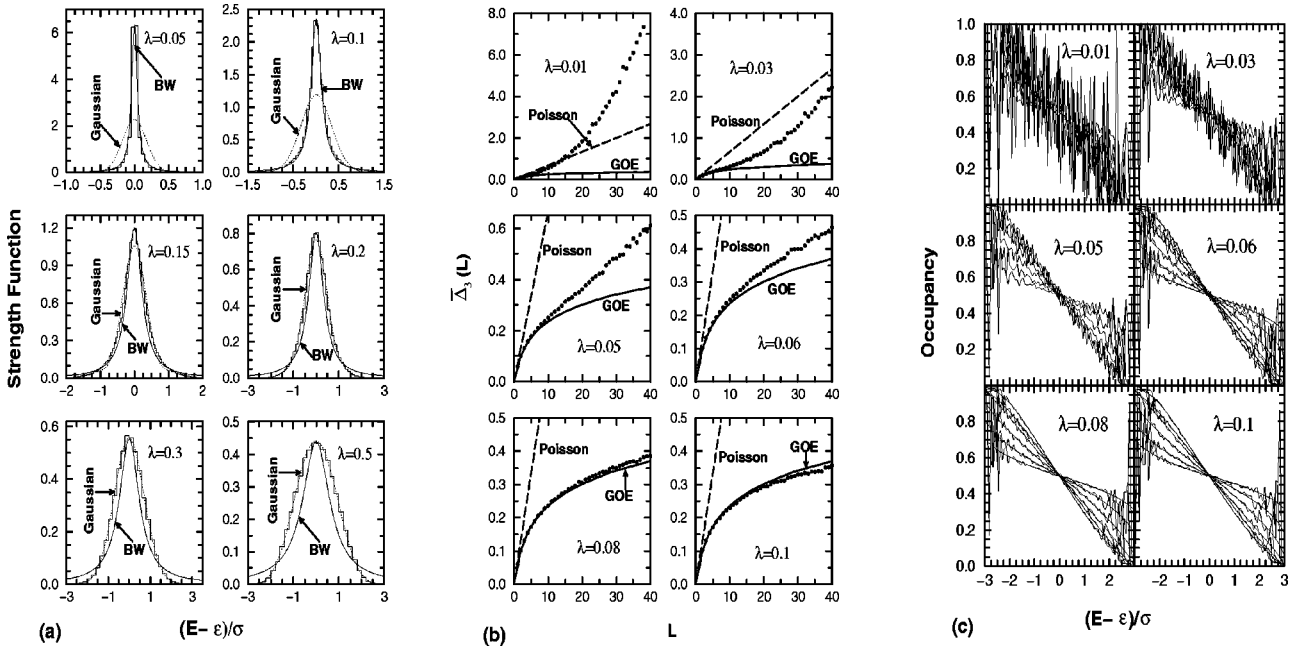

FIG. 1. Strength functions $F_k(E)$, Dyson-Mehta $\overline{\Delta}_3$ statistic for level fluctuations and occupancies $\langle E|n_i|E\rangle$ for EGOE(1+2) for various values of the interaction strength λ in $\{H\}=h(1)+\lambda\{V(2)\}$ for a system of 7 fermions in 14 single particle states (due to computational constraints, here, only one member is considered just as in [12]); the matrix dimension is 3432. The single-particle energies used in the calculations are $\epsilon_i=(i+1/i), i=1,2,\ldots,14$ just as in [12]. (a) The histograms are EGOE(1+2) results for the strength functions, continuous curves are BW fit and the dotted curves are Gaussian for $\lambda\leqslant 0.1$ and Edgeworth corrected Gaussian [16] for $\lambda>0.1$. In constructing the strength functions, $|C_k^E|^2$ are summed over the basis states $|k\rangle$ in the energy window $\hat{E}_k\pm\Delta$ and then the ensemble averaged $F_{\hat{E}_k}(\hat{E})$ vs $\hat{E}$ is constructed as a histogram; the value of Δ is chosen to be 0.025 for $\lambda\leqslant 0.1$ and beyond this $\Delta=0.1$. Here, $\hat{E}_k=(E_k-\epsilon_H)/\sigma_H$ and in the figure $\hat{E}_k=0$. Note that for $\lambda_{F_k}\sim 0.2$, there is BW to Gaussian transition. (b) The $\overline{\Delta}_3(L)$ statistic for overlapping intervals of length $L\leqslant 40$ are compared with Poisson and GOE values. For $\lambda\sim 0.06$, there is a Poisson to GOE transition in the $\overline{\Delta}_3$ statistic. (c) The wavy curves are numerical EGOE(1+2) results for occupancies and the smoothed curves with $\lambda\geqslant 0.06$ correspond to the results of EGOE(2) theory (ratio of Gaussians). Note that for $\lambda<0.06$, there are wide fluctuations in occupancies and the smoothed forms here are meaningless. All the results are shown for the lowest six single-particle states. Results similar to those in the figure, for the $N=12, m=6$ case, are reported in [1].

the normalized bivariate strength densities (matrix elements of $\mathcal{O}$ weighted by the state densities at the initial and final energies) $\rho_{biv;\mathcal{O}}(E_i,E_f)=[\langle\mathcal{O}^\dagger\mathcal{O}\rangle]^{-1}\langle\mathcal{O}^\dagger\delta(H-E_f)\mathcal{O}\delta(H-E_i)\rangle$ take bivariate Gaussian form for EGOE(2) and it is defined by the centroids (ϵ_i,ϵ_f) and widths (σ_i,σ_f) of its two marginals and the bivariate correlation coefficient that is given by $\langle\mathcal{O}^\dagger[(H-\epsilon_f)/\sigma_f]\mathcal{O}[(H-\epsilon_i)/\sigma_i]\rangle/\langle\mathcal{O}^\dagger\mathcal{O}\rangle$. Third, the level and strength fluctuations follow GOE. Moreover, with the Gaussian form for the state densities and bivariate Gaussian form for the strength densities, the strength sums $\langle E|\mathcal{O}^\dagger\mathcal{O}|E\rangle=\Sigma_{E'}|\langle E'|\mathcal{O}|E\rangle|^2$ take the form of ratio of two Gaussians, $\langle E|\mathcal{O}^\dagger\mathcal{O}|E\rangle=\langle\mathcal{O}^\dagger\mathcal{O}\rangle\rho_{\mathcal{O}^\dagger\mathcal{O};\mathcal{G}}(E)/\rho_{\mathcal{G}}(E)$ where $\rho_{\mathcal{O}^\dagger\mathcal{O};\mathcal{G}}(E)=\langle\mathcal{O}^\dagger\mathcal{O}\delta(H-E)\rangle$ is defined by its centroid $\epsilon_{\mathcal{O}^\dagger\mathcal{O}}=\langle\mathcal{O}^\dagger\mathcal{O}H\rangle/\langle\mathcal{O}^\dagger\mathcal{O}\rangle$ and variance $\sigma^2_{\mathcal{O}^\dagger\mathcal{O}}=\langle\mathcal{O}^\dagger\mathcal{O}H^2\rangle/\langle\mathcal{O}^\dagger\mathcal{O}\rangle-\epsilon^2_{\mathcal{O}^\dagger\mathcal{O}}$; $\mathcal{G}$ stands for Gaussian.

Hamiltonians for realistic interacting particle systems contain a mean-field part [one-body part $h(1)$] and a two-body residual interaction $V(2)$ mixing the configurations built out of the distribution of particles in the mean-field single-particle states; $h(1)$ is defined by the single-particle energies $\epsilon_i, i=1,2,\ldots,N$ and $V(2)$ is defined by its two-particle matrix elements. Then it is more realistic to use EGOE(1+2), the embedded Gaussian orthogonal ensemble of random matrices of (1+2)-body Hamiltonians where $\{H\}=h(1)+\lambda\{V(2)\}$; sometimes it is more convenient to use $\alpha h(1)+\lambda\{V(2)\}$. Here, $\{\}$ denotes ensemble, λ and α are free parameters and $V(2)$ in the two-particle spaces is a GOE with unit matrix elements variance; note that in general, $h(1)$ need not be fixed nor $V(2)$ a GOE (in this general, case, the ensemble is simply called the embedded ensemble of (1+2)-body interactions [EE(1+2)]; see [1] for more details). At this stage, it is important to stress that all the EGOE(2) results mentioned before are indeed applicable to EGOE(1+2) but only in the domain of chaos. Given (m,N) and the average spacing Δ [generated by $h(1)$] of the single-particle states (without loss of generality one can put $\Delta=1$) it is possible to find the critical λ value λ_c such that for $\lambda\geqslant\lambda_c$ there is the onset of chaos (GOE level fluctuations) in many ($m\gg 1$) particle spaces. In fact, λ_c is of the order of the spacing between m-particle mean-field basis states that are directly coupled by the two-body interaction; see the second and third reference in [4]. For $\lambda>\lambda_c$, for example, it is well established that the transition strength sums in EGOE(1+2) follow the EGOE(2) forms; see Fig. 1(c) ahead. References. [1,11] give many numerical examples for this, drawn from EGOE(1+2), and more importantly, for

atomic nuclei in several parts of the periodic table (detailed discussion of the nuclear physics examples is given in the last reference of [11]). It should be mentioned that the Gausian forms of state and transition strength densities are used in [10] to derive simple formulas for NPC and l_H in transition strength distributions.

For deriving formulas for NPC and l_H in wave functions, most useful quantity is the strength function (or local density of states) $F_k(E)$. Given the mean-field basis states $|k\rangle$ with energies $E_k=\langle k|H|k\rangle$, the eigenstates $|E\rangle$ can be expanded as $|E\rangle=\Sigma_k C_k^E|k\rangle$. Then the strength function $F_k(E)=\langle \delta(H-E)\rangle^k=\Sigma_{E'}|C_k^{E'}|^2\delta(E-E')$ and therefore it gives information about the structure of the eigenfunctions. In order to proceed further, let us say that the E_k energies are generated by a Hamiltonian H_k (the structure of H_k is discussed ahead). With this, it is easy to identify $F_k(E)$ as a conditional density of the bivariate density $\rho_{biv}(E,E_k)=\langle \delta(H-E)\,\delta(H_k-E_k)\rangle$. Taking degeneracies of E and E_k energies into account we have,

$$\rho_{biv}(E,E_k)=\langle \delta(H-E)\,\delta(H_k-E_k)\rangle=(1/d)\sum_{\alpha\in k,\beta\in E}|C_{k,\alpha}^{E,\beta}|^2$$

$$=(1/d)\overline{|C_k^E|^2}[d\rho^H(E)][d\rho^{H_k}(E_k)]$$

$$\Rightarrow$$

$$F_k(E)=\rho_{biv}(E,E_k)/\rho^{H_k}(E_k)$$

$$\overline{|C_k^E|^2}=\rho_{biv}(E,E_k)/[d\rho^H(E)\rho^{H_k}(E_k)]. \qquad (1)$$

In Eq. (1), d stands for the dimensionality of the m-particle spaces and $\overline{|C_k^E|^2}$ is the average of $|C_k^E|^2$ over all the degenerate states. Let us now examine the structure of H_k and $\rho_{biv}(E,E_k)$. First, it should be noted that the two-body interaction $V(2)$ can be decomposed into two parts $V(2)=V^{[0]}+\boldsymbol{V}$ so that $h(1)+V^{[0]}$ generates the E_k energies (diagonal matrix elements of H in the m-particle mean-field basis states). With m particles in N single-particle states, there is a $U(N)$ group and with respect to this group $V^{[0]}$ contains a scalar part $V^{[0],0}$ (a function of m), an effective (m-dependent) one-body (Hartree-Fock-like) part $V^{[0],1}$ and an irreducible two-body part $V^{[0],2}$. The $V^{[0],0}+V^{[0],1}$ will add to $h(1)$ giving an effective one-body part of H; $h(1)\rightarrow h(1)+V^{[0],0}+V^{[0],1}=\boldsymbol{h}$. The important point now being that, with respect to a $U(N)$ norm, the size of $V^{[0],2}$ is usually very small compared to the size of $\boldsymbol{h}$ in the m-particle spaces. With this, $H=\boldsymbol{h}+\boldsymbol{V}$ and then the H_k is nothing but $\boldsymbol{h}$. The piece $\boldsymbol{V}=V(2)-V^{[0]}$ generates the widths and other shape parameters of $F_k(E)$. It should be added that with respect to the $U(N)$ norm $\boldsymbol{h}$ and $\boldsymbol{V}$ are orthogonal and therefore $\sigma_H^2=\sigma_{\boldsymbol{h}}^2+\sigma_{\boldsymbol{V}}^2$. Definition of $V^{[0]}$, a brief discussion of its $U(N)$ decomposition, etc., are given in the Appendix. For EGOE(1+2), it is well known that the widths of $F_k(E)$ are in general constant; see [12] and the Appendix. The average variance of $F_k(E)$'s is given simply by

$$\overline{\sigma_k^2}=\sigma_{\boldsymbol{V}}^2=(d)^{-1}\sum_{\alpha\neq\beta}|\langle\alpha|H|\beta\rangle|^2,$$

where α and β are m-particle mean-field basis states indices. The results, (i) the norm of the $V^{[0],2}$ part is negligible and (ii) the widths of the strength functions are nearly constant (with little fluctuations) are well verified in a number of examples; see [13] and references in [1] for many nuclear physics examples. EGOE(1+2) discussions in the literature tacitly assume that $\boldsymbol{h}$ is $h(1)$ and $\boldsymbol{V}$ is $V(2)$ and the same is assumed from now on, i.e, $H=\boldsymbol{h}+\lambda\boldsymbol{V}\rightarrow h(1)+\lambda V(2)$. In addition to (i) and (ii), it is well verified in a number of numerical calculations that: (iii) $F_k(E)$'s exhibit a transition from BW to Gaussian form in the chaotic domain defined by $\lambda>\lambda_{F_k}$; usually $\lambda_c<\lambda_{F_k}$; see [1,14] for some analytical understanding of this result. The results (i), (ii), and (iii) clearly imply that the $\rho_{biv}(E,E_k)$ is a bivariate Gaussian and this result was first mentioned in [15]. A numerical example for the BW to Gaussian transition in strength functions in EGOE(1+2) is shown in Fig. 1(a). In this example $\lambda_{F_k}\sim 0.2$ and it is much larger than $\lambda_c=0.06$ obtained via the results for the Dyson-Mehta $\overline{\Delta}_3$ level statistic shown in Fig. 1(b). Thus, there is the onset of GOE fluctuations much before the $F_k(E)$'s start taking Gaussian form, i.e., $\lambda_{F_k}>\lambda_c$. Unlike the case with strength functions (also transition strength densities; see [17]), as mentioned before, strength sums start following the EGOE(2) form (i.e., ratio of Gaussians) from $\lambda=\lambda_c$. This is demonstrated in Fig. 1(c) where occupancies $\langle E|n_i|E\rangle$ as a function of E are shown [they correspond to the strength sums generated by single state ($|i\rangle$) destruction operators]. As mentioned in the Introduction, the nature of NPC (which is the inverse of IPR) in wave functions in the $\lambda_c\leqslant\lambda<\lambda_{F_k}$ domain where $F_k(E)$ is of BW form (i.e., in the BW domain) was studied in [9] while the present article is concerned with the $\lambda>\lambda_{F_k}$ domain (i.e., the Gaussian domain) where $F_k(E)$ is of Gaussian form.

III. EGOE(1+2) RESULTS FOR NPC AND l_H IN WAVE FUNCTIONS

For EGOE(1+2), in the chaotic domain with $\lambda>\lambda_{F_k}$, one has from Sec. II the results: (i) E_k are generated by $H_k=h(1)$, therefore the variance of $\rho^{H_k}(E_k)$ is $\sigma_{\boldsymbol{h}}^2$; (ii) widths of the strength functions are constant and they are generated by $V(2)$, the average variance $\overline{\sigma_k^2}=\sigma_{\boldsymbol{V}}^2$; (iii) $F_k(E)$'s are Gaussian in form; (iv) $F_k(E)$ is a conditional density of the bivariate Gaussian $\rho_{biv:\mathcal{G}}(E,E_k)$. The correlation coefficient ζ of $\rho_{biv:\mathcal{G}}(E,E_k)$ is given by,

$$\zeta=\frac{\langle(H-\epsilon_H)(H_k-\epsilon_H)\rangle}{\sqrt{\langle(H-\epsilon_H)^2\rangle\langle(H_k-\epsilon_H)^2\rangle}}=\sqrt{\left(1-\frac{\overline{\sigma_k^2}}{\sigma_H^2}\right)}. \qquad (2)$$

Note that the centroids of the E and E_k energies are both given by $\epsilon_H=\langle H\rangle$. In Eq. (2), the second equality is obtained by using the orthogonality between $h(1)$ and $V(2)$ operators. It is immediately seen that the ζ^2 is nothing but the

variance of E_k's [the centroids of $F_k(E)$] normalized by the state-density variance. The $\rho_{biv:\mathcal{G}}(E,E_k)$, which takes into account the fluctuations in the centroids of $F_k(E)$ and assumes that the variances are constant, is used to derive formulas for NPC and l_H in the wave functions (methods of taking into account variance fluctuations will be discussed ahead) $\psi_E=|E\rangle$ expanded in the mean-field basis defined by the states $\phi_k=|k\rangle$. Let us first define NPC and l_H,

$$|E\rangle=\sum_k C_k^E|k\rangle$$

$$\Rightarrow$$

$$(\mathrm{NPC})_E=\left[\sum_k \left|C_k^E\right|^4\right]^{-1},$$

$$l_H(E)=\exp[(S^{info})_E]/(0.48\,d),$$

$$(S^{info})_E=-\sum_k |C_k^E|^2\ln|C_k^E|^2. \tag{3}$$

In Eq. (3) $0.48d$ is the GOE value for S^{info}, thus, $l_H=1$ for GOE. Similarly, NPC is $d/3$ for GOE.

In terms of the locally renormalized amplitudes $\mathcal{C}_k^E=C_k^E/\sqrt{\overline{|C_k^E|^2}}$ where the bar denotes ensemble average with respect to EGOE(1+2), $\Sigma_k\overline{|C_k^E|^4}=\Sigma_k\overline{|\mathcal{C}_k^E|^4}(\overline{|C_k^E|^2})^2$. Then the ensemble averaged $(\mathrm{NPC})_E$ is obtained as follows:

$$\overline{\sum_k |C_k^E|^4} \overset{EGOE(1+2)}{\rightarrow} \sum_k \overline{|\mathcal{C}_k^E|^4}(\overline{|C_k^E|^2})^2$$

$$=3\sum_k (\overline{|C_k^E|^2})^2$$

$$=\frac{(3/d)}{[\rho_{\mathcal{G}}^H(E)]^2}\int dE_k\frac{[\rho_{biv:\mathcal{G}}(E,E_k)]^2}{\rho_{\mathcal{G}}^{H_k}(E_k)}$$

$$=\frac{(3/d)}{[\rho_{\mathcal{G}}^H(E)]^2}\int dE_k\rho_{\mathcal{G}}^{H_k}(E_k)[F_{k:\mathcal{G}}(E)]^2$$

$$\Rightarrow$$

$$(\mathrm{NPC})_E=(d/3)\sqrt{1-\zeta^4}\exp-\left\{\frac{\zeta^2\hat{E}^2}{1+\zeta^2}\right\}. \tag{4}$$

The $\hat{E}$ in Eq. (4) is the standardized E, i.e., it is zero centered and normalized to unit width, $\hat{E}=(E-\epsilon_H)/\sigma_H$. In the first step in Eq. (4), the fact that EGOE exhibits average fluctuations separation (with little communication between the two) is used. For example, in the normal mode decomposition of the EGOE state density, it is seen that the long wave-length parts generate the smoothed Gaussian density (with corrections) and the short wave-length parts the GOE fluctuations with damping of the intermediate ones (see [18–20] for detailed discussions on this important result). This allows one to carry out $|\mathcal{C}_k^E|^4$ ensemble average independent of the other smoothed (average) term. In the second line, the Porter-Thomas form of local strength fluctuations is used and then $\overline{|\mathcal{C}_k^E|^4}=3$, a GOE result. In the third step, the result in Eq. (1) is used. Then, the Gaussian forms, valid in the chaotic domain ($\lambda>\lambda_{F_k}$), of all the densities for EGOE(1+2) give the final formula (this result was quoted first in [10] without details). Before turning to the formula for the localization length l_H, let us briefly discuss about the corrections to Eq. (4) due to the fluctuations in the variances of $F_k(E)$; the form with $F_k(E)$ shown explicitly, is written in Eq. (4) for this purpose and this form also allows one to understand the results in [21] as discussed ahead.

The correction to NPC due to $\delta\sigma_k^2=\sigma_k^2-\overline{\sigma_k^2}\neq 0$ is obtained by using, for small $|\delta\sigma_k^2|$, the hermite polynomial expansion which gives [16], $F_{k:\mathcal{G}}(E)\rightarrow F_{k:\mathcal{G}}(E)\{1+c_2(\mathcal{E}_k^2-1)\}$ where $c_2=\delta\sigma_k^2/2\overline{\sigma_k^2}$ and $\mathcal{E}_k=(E-E_k)/\sqrt{\overline{\sigma_k^2}}$. This corrected $F_k(E)$ is used in the integral form with $F_k(E)$ in Eq. (4). As NPC involves sum over all the $|k\rangle$ states, it is a valid assumption to treat $(\delta\sigma_k^2)$'s as random and therefore in $[F_k(E)]^2$ only the terms that are quadratic in $(\delta\sigma_k^2)$ will contribute (see [21]). Replacing $[(\delta\sigma_k^2)/\overline{\sigma_k^2}]$ by $(\delta\sigma^2)/\overline{\sigma_k^2}=[(d)^{-1}\{\Sigma_k(\delta\sigma_k^2)^2\}]^{1/2}/\overline{\sigma_k^2}$ and substituting the corrected $F_k(E)$ for $F_{k:\mathcal{G}}(E)$ in Eq. (4), we get

$$(\mathrm{NPC})_E=\frac{(3/d)}{[\rho_{\mathcal{G}}^H(E)]^2}\int dE_k\rho_{\mathcal{G}}^{H_k}(E_k)[F_{k:\mathcal{G}}(E)]^2$$

$$\times\left(1+\frac{(\delta\sigma^2)}{2\overline{\sigma_k^2}}(\mathcal{E}_k^2-1)\right)^2$$

$$=\frac{d}{3}\sqrt{1-\zeta^4}\exp-\left\{\frac{\zeta^2\hat{E}^2}{1+\zeta^2}\right\}$$

$$\times\left\{1+\frac{1}{4}\left[\frac{(\delta\sigma^2)}{\sigma_H^2}\right]^2X(E)\right\}^{-1}; \tag{5}$$

$$X(E)=\frac{1}{(1+\zeta^2)^4}\left[\hat{E}^4-2\frac{(1+\zeta^2)(1-2\zeta^2)}{1-\zeta^2}\hat{E}^2\right.$$

$$\left.+\left(\frac{1+\zeta^2}{1-\zeta^2}\right)^2(1+2\zeta^4)\right].$$

The $\delta\sigma^2$ correction term in Eq. (5) is valid only when the fluctuations in the variances of $F_k(E)$'s are small (this is in general always true). For small ζ values, the formula for NPC in Eq. (5) reduces to the expression given recently, for EGOE(2), by Kaplan and Papenbrock [21]; they use ideas related to ''scar theory.'' In the EGOE(1+2) Hamiltonian $H=h(1)+\lambda V(2)$, with $\lambda\rightarrow\infty$ one obtains EGOE(2) and then it is clear from the definition in Eq. (2) that in this limit $\zeta\sim 0$. More precisely, with $N\gg m\gg 1$, $\zeta^2\sim\binom{N}{2}^{-1}$ and $[(\delta\sigma^2)/\sigma_H^2]^2\sim[\binom{m}{2}\binom{N}{2}]^{-1}$ for $\{H\}=\{V(2)\}$; see the Appendix. Therefore for finite N, the correlation coefficient and the

variance corrections are small but nonzero and in the large N limit, they are zero giving the GOE result as pointed out in [10]. As we add the mean-field part to the EGOE(2), ζ increases and at the same time the variance correction decreases; see the Appendix. Thus, the formula (5) with the $(\delta\sigma^2)$ term is important only for small ζ. Equation (4) is accurate for reasonably large ζ (say for $\zeta \geqslant 0.3$) as in the examples discussed in [10]. All these results are well tested by the numerical examples in Sec. IV.

Proceeding exactly as in Eq. (4), formula for the localization length l_H as a function of the excitation energy is derived. Using the definition (3), writing $|C_k^E|^2$ in terms of $\overline{|C_k^E|^2}$ and $\overline{|C_k^E|^2}$, using the GOE results $\overline{|C_k^E|^2}=1$ and $\overline{|C_k^E|^2 \ln(|C_k^E|^2)} = -\ln 0.48$, applying the last equality in Eq. (1) and replacing all the densities by their corresponding Gaussian forms, converting the sum in Eq. (3) into an integral and finally carrying out the integration, the expression for l_H in wave functions is obtained,

$$l_H(E) \overset{EGOE(1+2)}{\rightarrow} -\int dE_k \frac{\rho_{biv:\mathcal{G}}(E,E_k)}{\rho_{\mathcal{G}}^H(E)} \ln\left\{ \frac{\rho_{biv:\mathcal{G}}(E,E_k)}{\rho_{\mathcal{G}}^{H_k}(E_k)\rho_{\mathcal{G}}^H(E)} \right\}$$

$$= \sqrt{1-\zeta^2}\exp\left(\frac{\zeta^2}{2}\right)\exp-\left(\frac{\zeta^2\hat{E}^2}{2}\right). \qquad (6)$$

The result in Eq. (6) was reported in [10] without details. By rewriting the integral in Eq. (6) in terms of $F_k(E)$ and making small $(\delta\sigma^2)$ expansion just as in the case of NPC, the formula incorporating corrections due to fluctuations (with respect to k) in the variances of $F_k(E)$ is derived following the arguments that led to Eq. (5). Neglecting higher-order terms in $[(\delta\sigma^2)/\sigma_H^2]^2$, the final result is

$$l_H(E) = \sqrt{1-\zeta^2}\exp\left(\frac{\zeta^2}{2}\right)\exp\left[-\left(\frac{\zeta^2\hat{E}^2}{2}\right)\right]$$

$$\times\left(1-\frac{1}{8}\left[\frac{(\delta\sigma^2)}{\sigma_H^2}\right]^2 Y(E)\right); \qquad (7)$$

$$Y(E) = \frac{1}{(1-\zeta^2)^2}\{(1-\zeta^2)^2(\hat{E}^2-1)^2 + 4\zeta^2(1-\zeta^2)$$

$$\times\hat{E}^2 + 2\zeta^4\}.$$

IV. NUMERICAL TESTS

NPC and l_H are calulated for a EGOE(1+2) with 6 particles in 12 single-particle states and the results are compared with Eqs. (4–7) in Fig. 2. In the numerical calculations, the single-particle energies $(i+1/i), i=1,2,\ldots,12$ define $h(1)$ (as in [12] and Fig. 1), in the two-particle space $V(2)$ is a GOE (calculations use 25 members) with unit matrix elements variance and the Hamiltonian ensemble is $\{H_{\alpha,\lambda}\} = \alpha h(1) + \lambda\{V(2)\}$. The value of $\lambda = 0.2$ is fixed so that, for $\alpha \leqslant 1$ the level fluctuations are of GOE; i.e., one is in the chaotic domain (see [1,17] and Fig. 1). Results for $\alpha = 0, 0.5$, and 1 in Figs. 2(a) and 2(b) clearly demonstrate that

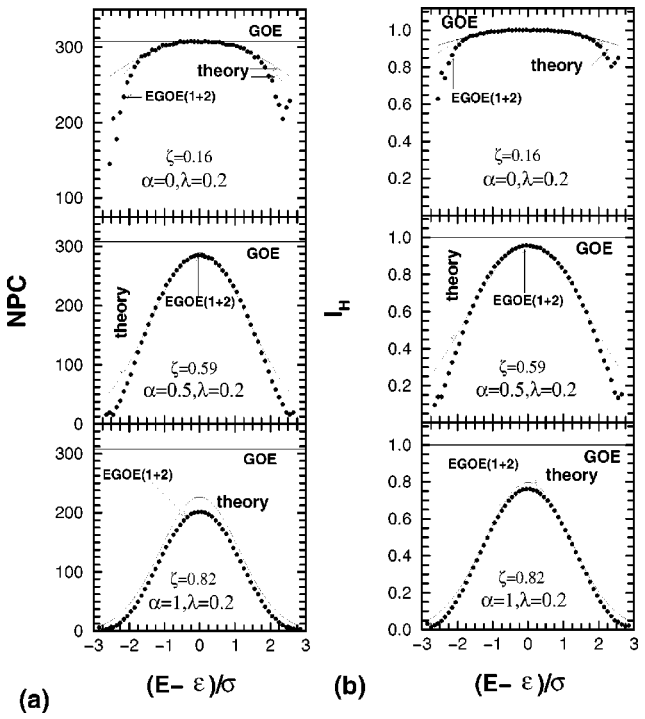

FIG. 2. (a) Number of principal components NPC and (b) the localization length l_H in wave functions for a system of 6 interacting particles in 12 single-particle states (matrix dimension is 924). Here, for conveniance, the EGOE(1+2) Hamiltonian is changed to $\{H_{(\alpha,\lambda)}\} = \alpha h(1) + \lambda\{V(2)\}$. Numerical EGOE(1+2) results correspond to filled circles. The continuous curves correspond to the theory (4) for NPC and Eq. (6) for l_H. For the case with $\alpha = 0$, the dashed curves correspond to the theory (5) for NPC and Eq. (7) for l_H. For other cases, the correction due to variance fluctuations is negligible, and hence only the results of Eqs. (4) and (6) are shown in the figure. Note that NPC$=d/3$ and $l_H = 1$ for GOE. See text for further details.

the EGOE(1+2) formulas based on the bivariate Gaussian form for $\rho_{biv}(E,E_k)$ are excellent. In these examples ζ values are 0.16, 0.59, and 0.82, respectively. The $(\delta\sigma^2)$ correction is seen to be important only for the case with $\alpha = 0$. In fact, the $[(\delta\sigma^2)/\sigma_H^2]^2$ values for the three cases considered are 0.121×10^{-1}, 0.545×10^{-2}, and 0.134×10^{-2}. Thus, for realistic fermion models that are represented by EGOE(1+2) (with $\lambda > \lambda_{F_k}$), the correction due to variance fluctuations is expected to be significant only in the situation ζ is small. Extension of EGOE(2) with explicit inclusion of spin degrees of freedom (each single-particle level is taken to be doubly degenerate with $s_z = \pm 1/2$; see the third reference in [4]) was considered and for a system of six fermions in seven levels (i.e., $m=6$, $N=7\times2$) with total $S_z = 0$, giving $d = 1225$, NPC was calculated as a function of the excitation energy in [21]; we call this model EGOE(2)-S. In this example, as given in [21], $\zeta = 0.3$ and $[(\delta\sigma^2)/\sigma_H^2]^2 = 0.052$. Thus, here the corrections due to variance fluctuations are non-negligible (the situation in this case is similar to the $\alpha = 0$ case in Fig. 2) and applying Eq. (5) gives an excellent description, as shown in Fig. 3(a), of the results for NPC reported in [21] for the EGOE(2)-S model. Returning to Fig.

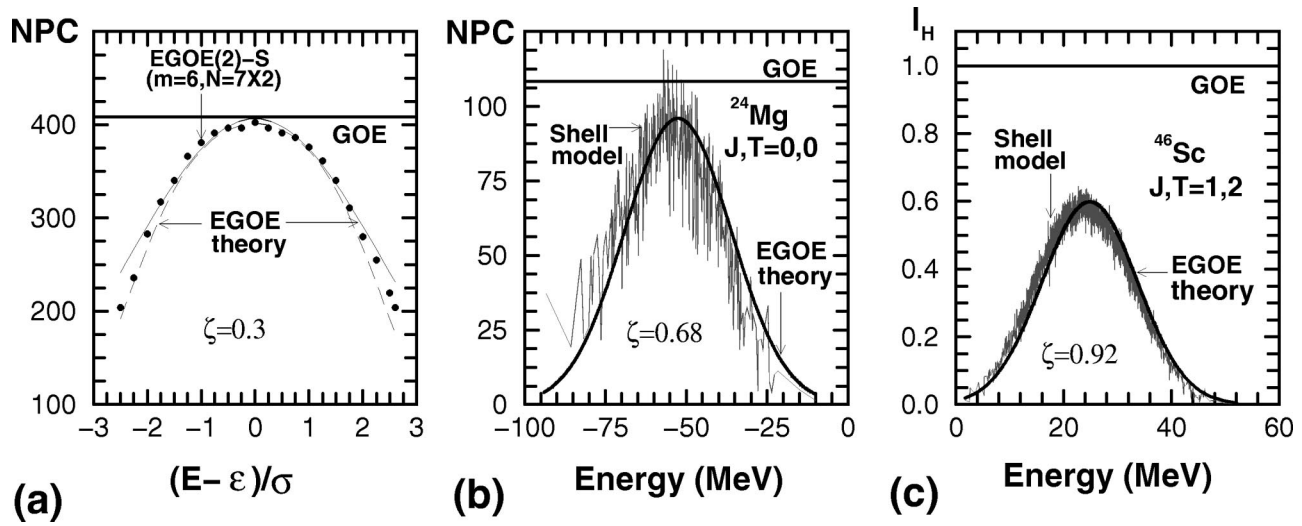

FIG. 3. (a) NPC for the EGOE(2)-S model described in the text compared with the results given by Eqs. (4) and (5). The filled circles are for the numerical EGOE(2)-S calculations reported in [21]. The continuous and dashed curves represent Eqs. (4) and (5), respectively. (b) NPC for the (sd) shell nucleus ^{24}Mg compared with Eq. (4). The shell-model calculations are the same as in [11]. In this example, Eqs. (4) and (5) give almost identical results, and hence, the curve corresponding to Eq. (5) is not shown in the figure. (c) Shell-model results for ^{46}Sc for l_H reported in [22] compared with the theoretical curve given by Eq. (6) with $\zeta=0.92$.

2, it should be mentioned that there are differences between the numerical results and the predictions based on Eqs. (4) and (6) even for the cases with $\zeta=0.59$ and 0.82. These may be due to the departures of $\rho_{biv}(E,E_k)$ from the bivariate Gaussian form. An important observation from Eqs. (4) and (6) is, at the spectrum center NPC$=(d/3)\sqrt{1-\zeta^4}$ and $l_H=\sqrt{1-\zeta^2}\exp(\zeta^2/2)$. Therefore for ζ^2 close to 0.8 or large, there will be large deviations from GOE even at the spectrum center for a system described by EGOE$(1+2)$. This is clearly seen in the $\alpha=1$ case in Fig. 2; here $\zeta=0.82$. Finally, it should be mentioned that the EGOE$(1+2)$ calculations for the $N=14$, $m=7$ system (the case considered in Fig. 1) are also carried out and the results are seen to be essentially same as in Figs. 2(a,b).

Let us now turn to the nuclear shell model that is a realistic interacting fermion model. There are shell-model results for the $(2s1d)$ shell (here after called sd shell) nuclei ^{28}Si [3] and ^{22}Na (see [1] and the second reference in [11]) for NPC and l_H in wave functions. For ^{28}Si the 839 dimensional $J=0,T=0$ space (with six protons and six neutrons in the sd shell) and the 3243 dimensional $J=2,T=0$ space are considered. Similarly, for ^{22}Na the 307 dimensional $J=2,T=0$ space (with three protons and three neutrons in the sd shell) is considered. The results for these nuclei are analyzed using Eqs. (4) and (6) as briefly discussed in [10,1]. In all the sd shell examples, $\zeta\sim0.6-0.7$ and therefore the situation is similar to the $\alpha=0.5$ case in Fig. 2. Thus, in these examples, the departures from GOE at the spectrum center are no more than 10% but away from the center, there are large departures. The shell model NPC and l_H for sd shell nuclei are seen to be well described by the EGOE forms in Eqs. (4) and (6). For further confirming this, NPC is evaluated for ^{24}Mg in the 325 dimensional $J=0,T=0$ space (with four protons and four neutrons in the sd shell) and the results are shown in Fig. 3(b); here $\zeta=0.68$. It can be concluded that the deviations of the sd shell-model results from GOE clearly imply that the shell-model Hamiltonians are well represented by EGOE$(1+2)$ (with $\lambda>\lambda_{F_k}$) but not by GOE. It is also seen that the corrections due to $(\delta\sigma^2)$ are small for (sd) shell nuclei (note that here ζ is large); in the ^{24}Mg example, $[(\delta\sigma^2)/\sigma_H^2]^2=0.024$. In order to further substantiate the EGOE description of the structure of nuclear shell-model wave functions, we have analyzed using Eq. (6) the $l_H(E)$ vs E results reported recently in [22] for $2p1f$ shell (hereafter called pf shell) nuclei ^{50}Ca and ^{46}Sc. In the case of ^{50}Ca the 2051-dimensional $J=6,T=5$ space (with ten protons in the pf shell) and in ^{46}Sc the 2042 dimensional $J=1,T=2$ space (with one proton and five neutrons in the pf shell) are considered and a modern large shell-model code was used for obtaining the l_H values. The shell-model results for l_H in Fig. 9 of [22], via Eq. (6), determine ζ to be 0.96 and 0.92, respectively, for the ^{50}Ca and ^{46}Sc examples; results for ^{46}Sc are shown in Fig. 3(c). From the definition (2) but employing averages over mT spaces (instead of mJT spaces), we obtain the ζ values 0.91 and 0.89, respectively. It should be pointed out that given the single-particle energies and the two-body matrix elements of the shell-model Hamiltonians, it is easy to calculate ζ in fixed mT spaces using trace propagation methods [by extending Eqs. (A3) and (A4)] [1,15]. The pf shell examples are similar to the $\alpha=1$ case in Fig. 2 and therefore, as expected, one sees large departures from GOE even at the spectrum center. Finally, it is seen from the shell-model examples in Fig. 3 and the EGOE examples in Fig. 2 that further corrections to the results in Eqs. (4)–(7) need to be worked out but this is not attempted in this paper. Similarly, study of the nature of fluctuations in NPC and l_H is postponed for the future.

V. CONCLUSIONS

Wave-function structure given by the EGOE$(1+2)$ random matrix ensemble $\{H\}=h(1)+\lambda\{V(2)\}$ is studied by

deriving compact formulas for NPC and l_H. They are based on: (i) the Gaussian form for strength functions $F_k(E)$'s and the bivariate Gaussian form for $\rho_{biv}(E,E_k)$ [with $F_k(E)$ being a conditional density of $\rho_{biv}(E,E_k)$] which are valid in the chaotic domain defined by $\lambda > \lambda_{F_k}$; (ii) there is average-fluctuations separation (with little communication between the two) in energy levels and strengths with local strength fluctuations following the Porter-Thomas law; (iii) there is a significant unitary group decomposition of the hamiltonian. With EGOE(1+2), the NPC and l_H take Gaussian forms as a function of the excitation energy and they are defined by the bivariate correlation coefficient ζ which measures the variance of the distribution of centroids of $F_k(E)$'s relative to the state-density variance. Theory for incorporating corrections due to fluctuations in the variances (with k) of $F_k(E)$ is also given. For small ζ, the present formulation gives back the results for pure EGOE(2) [i.e., in the $\lambda \to \infty$ limit of EGOE(1+2)] as derived in [21] recently. The formulas derived for NPC and l_H are subjected to numerical EGOE(1+2) tests with ζ changing from 0.1 to 0.8. These and the analysis of the results for a EGOE(2)-*S* example and some nuclear shell-model examples, clearly point out that isolated finite realistic interacting particle systems, in the chaotic domain ($\lambda \geqslant \lambda_{F_k}$), will have wave-function structure as given by EGOE(1+2). Finally, in the theory given by Eqs. (4) and (6), NPC and l_H depend on just one parameter and this appears to be an aspect of ''geometric chaos'' (see [23] for a recent discussion on geometric chaos).

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APPENDIX

Let us consider a system of m fermions in N single-particle states with a (1+2)-body Hamiltonian $H=h(1)+V(2)$ where $h(1)$ is specified by the single-particle energies ϵ_i [with i denoting the ith single-particle state, $h(1)=\Sigma_i \epsilon_i n_i$ where n_i are number operators] and $V(2)$ by the two-body matrix elements $V_{ijkl}=\langle kl|V(2)|ij\rangle$. The two-body interaction can be seperated into $V(2)=V^{[0]}+\boldsymbol{V}$ where $V^{[0]}$ is given by

$$V^{[0]}=\sum_{i<j} V_{ijij} n_i n_j . \tag{A1}$$

The $h(1)+V^{[0]}$ generates the $F_k(E)$ centroids E_k. With N single-particle states, there is a $U(N)$ group generated by the N^2 operators $a_i^\dagger a_j$ where $a_i^\dagger$ and a_j are one-particle creation and destruction operators, respectively. With respect to this $U(N)$ group, $V^{[0]}$ decomposes into $\nu=0,1,2$ parts and their explicit structure is (for a given m),

$$V^{[0],0}=\binom{m}{2} V^0; \quad V^0=\binom{N}{2}^{-1} \sum_{i<j} V_{ijij},$$

$$V^{[0],1}=\frac{m-1}{N-2} \sum_i \zeta_i n_i; \quad \zeta_i=\sum_{j\neq i} (V_{ijij}-V^0),$$

$$V^{[0],2}=V^{[0]}-V^{[0],0}-V^{[0],1}. \tag{A2}$$

Similarly, the $h(1)$ operator will have $\nu=0,1$ parts; $h^0=m\epsilon^0$ where $\epsilon^0=(N)^{-1}\Sigma_i \epsilon_i$ and $h^1(1)=\Sigma_i \epsilon_i^1 n_i$ where $\epsilon_i^1=\epsilon_i-\epsilon^0$. Finally it is to be noted that $\boldsymbol{V}$ behaves essentially as a $\nu=2$ operator (the $\nu=1$ part of $\boldsymbol{V}$ is of negligible size in the $N \gg m \gg 1$ limit).

The $U(N)$ norm (in the m-particle spaces) of an operator $\mathcal{O}$ is defined by $||\mathcal{O}||_m=\sqrt{\langle (\mathcal{O}-\langle\mathcal{O}\rangle^m)^\dagger (\mathcal{O}-\langle\mathcal{O}\rangle^m)\rangle^m}$. An important theorem is that the $\nu=0,1,2$ parts of H are orthogonal with respect to this $U(N)$ norm. For a $\nu=1$ operator $\mathcal{O}(1)=\Sigma_i e_i n_i$, the norm square is simply given by

$$||\mathcal{O}(1)||_m^2=\frac{m(N-m)}{N(N-1)} \sum_i e_i^2 . \tag{A3}$$

Similarly for a $\nu=2$ operator $\mathcal{O}(2)$,

$$||\mathcal{O}(2)||_m^2=\frac{m(m-1)(N-m)(N-m-1)}{2(N-2)(N-3)} \langle \mathcal{O}^\dagger(2)\mathcal{O}(2)\rangle^2. \tag{A4}$$

Using Eqs. (A3) and (A4) one can calculate the norms of $h^1+V^{[0],1}$ and $V^{[0],2}$ and in general the later is very small compared to the former. Then $h(1)+V^{[0]} \to \boldsymbol{h}=\Sigma_i \xi_i n_i$ where $\xi_i=\epsilon_i^1+(m-1/N-2)\zeta_i$ (note that at the end we add the spectrum centroid generating part $h^0+V^{[0],0}$ to $\boldsymbol{h}$). Thus, neglecting the $V^{[0],2}$ part, the centroids of $F_k(E)$'s are generated by $\boldsymbol{h}$ and the variances by $\boldsymbol{V}$. As $\boldsymbol{h}$ and $\boldsymbol{V}$ are orthogonal, $\sigma_H^2=\sigma_{\boldsymbol{h}}^2+\sigma_{\boldsymbol{V}}^2$. These variances, in m-particle spaces, follow easily from Eqs. (A3) and (A4). See [1,24] for further details.

Let us consider a EGOE(1+2) Hamiltonian $H=\alpha h(1)+\lambda V(2)$ with unit spacing between the ϵ_i's and the V_{ijkl} taken as zero-centered Gaussian variables with unit variance. In the $N>>m>>1$ situation one can study the behavior of ζ^2 and $(\delta\sigma^2)$ as follows. The correlation coefficient $\zeta^2=\sigma_{\boldsymbol{h}}^2/\sigma_H^2$ and, neglecting the contributions of $V(2)$ to $\sigma_{\boldsymbol{h}}$, one gets $\sigma_{\boldsymbol{h}}^2 \sim (mN^2/12)\alpha^2$. Similarly, $\sigma_{\boldsymbol{V}}^2 \sim \binom{m}{2}\binom{N}{2}\lambda^2$. Here, Eqs. (A3) and (A4) are used. Therefore, $\zeta^2=[(1+3m\,(\lambda/\alpha)^2]^{-1}$ and this expression gives 0.51 and 0.76 for the $\alpha=0.5$ and 1 cases in Fig. 2. They compare well with the exact numbers given in Fig. 2. However, this estimate fails in the situation $\alpha \to 0$. For $\alpha=0$, the $\boldsymbol{h}$ has to be replaced by $V^{[0]}$ and then the E_k energies are a sum of $\binom{m}{2}$ zero-centered Gaussian variables each with variance λ^2. This, together with the $\sigma_{\boldsymbol{V}}^2$ expression, gives $\zeta^2 \sim \binom{m}{2}^{-1}$ for $\alpha \sim 0$ as pointed out in [21]. The number quoted for the $\alpha=0$ case in Fig. 2 is close to this estimate. Finally, an estimate for $[(\delta\sigma^2)/\sigma_H^2]^2$ is obtained from Eq. (A4) by noting that $\sigma_{\boldsymbol{V}}^2$ is a sum of $K \sim \binom{m}{2}\binom{N}{2}$ χ^2-variables and therefore $[(\delta\sigma^2)/\sigma_{\boldsymbol{V}}^2]^2=2/K$ as given first in [12]. Then, $\sigma_{\boldsymbol{V}}^2=(1-\zeta^2)\sigma_H^2$ gives the final result $[(\delta\sigma^2)/\sigma_H^2]^2 \sim 2(1-\zeta^2)/\binom{m}{2}\binom{N}{2}$.

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